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TWO-SIDED IDEALS IN LEAVITT PATH ALGEBRAS

PINAR COLAK
Department Of Mathematics, Simon Fraser University
8888 University Drive, Burnaby, BC V5A 1S6, Canada
ppekcagl@sfu.ca

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We explicitly describe two-sided ideals in Leavitt path algebras associated with a row-finite graph. Our main result is that any two-sided ideal I of a Leavitt path algebra associated with a row-finite graph is generated by elements of the form $v + \sum_{i=1}^{n} \lambda_i g^i$, where g is a cycle based at vertex v. We use this result to show that a Leavitt path algebra is two-sided Noetherian if and only if the ascending chain condition holds for hereditary and saturated closures of the subsets of the vertices of the row-finite graph E.

Keywords: Leavitt path algebra; two-sided Noetherian; two-sided ideal

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Throughout this paper K denotes a field. For a directed graph E, the Leavitt path algebra $L_K(E)$ of E with coefficients in K has received much recent attention, see e.g. [1], [5], [6]. The two-sided ideal structure of $L_K(E)$ has been an important focus of much of this work. In this paper we provide an explicit description of a generating set for any two-sided ideal of $L_K(E)$, where E is any row-finite graph. We then use this description to identify those row-finite graphs E for which $L_K(E)$ is two-sided Noetherian.

We briefly recall the basic definitions.

A *directed graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0, E^1 and functions $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and elements of E^1 are called *edges*. For each $e \in E^1$, $r(e)$ is the *range* of e and $s(e)$ is the *source* of e. If $r(e) = v$ and $s(e) = w$, then we say that v *emits* e and that w *receives* e. A vertex which emits no edges is called a *sink*. A graph is called *row-finite* if every vertex is the source of at most finitely many edges.

A *path* μ in a graph E is a sequence of edges $\mu = e_1 \cdots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n-1$. We define the source of μ by $s(\mu) := s(e_1)$ and the range of μ by $r(\mu) := r(e_n)$. If we have $r(\mu) = s(\mu)$ and $s(e_i) \neq s(e_{i+1})$ for every

$i \neq j$, then μ is called a *cycle*. A *closed path based at* v is a path $\mu = e_1 \cdots e_n$, with $e_j \in E^1$, $n \geq 1$ and such that $s(\mu) = r(\mu) = v$. We denote the set of all such paths by $CP(v)$. A *closed simple path based at* v is a closed path based at v, $\mu = e_1 \dots e_n$, such that $s(e_j) \neq v$ for $j > 1$. We denote the set of all such paths by $CSP(v)$. Note that a cycle is a closed simple path based at any of its vertices. However the converse may not be true, as a closed simple path based at v may visit some of its vertices (but not v) more than once.

Let v be a vertex in E^0. If there is no cycle based at v, then we let $g = v$ and call it the *trivial* cycle. If g is a cycle based at v of length at least 1, then g is called a *non-trivial* cycle.

Let $E = (E^0, E^1)$ be any directed graph, and let K be a field. We define the *Leavitt path K-algebra* $L_K(E)$ associated with E as the K-algebra generated by a set $v \in E^0$ together with a set $\{e, e^* | e \in E^1\}$, which satisfy the following relations:

(1) $vv' = \delta_{v,v'} v$ for all $v, v' \in E^0$.
(2) $s(e)e = er(e) = e$ for all $e \in E^1$.
(3) $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$.
(4) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in E^1$.
(5) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for every $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$.

The elements of E^1 are called *real edges*, while for $e \in E^1$ we call e^* a *ghost edge*. The set $\{e^* | e \in E^1\}$ is denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. We say that a path in $L_K(E)$ is a *real path* (resp., a *ghost path*) if it contains no terms of the form e_i^* (resp., e_i). We say that $p \in L_K(E)$ is a polynomial in *only real edges* (resp., in *only ghost edges*) if it is a sum of real paths (resp., ghost edges). The *length* of a real path (resp., ghost path) μ, denoted by $|\mu|$, is the number of edges it contains. The length of $v \in E^0$ is 0. Let x be a polynomial in only real edges (resp., in only ghost edges) in $L_K(E)$. If $x = \mu_1 + \cdots + \mu_n$, where the μ_i's are real paths (resp., ghost paths), then the *length* of x, denoted by $|x|$, is defined as $\max_{i=1,\dots,n}\{|\mu_i|\}$. An edge e is called an *exit* to the path $\mu = e_1 \cdots e_n$ if there exists i such that $s(e) = s(e_i)$ and $e \neq e_i$.

The proofs of the following can be found in [1].

Lemma 1. *$L_K(E)$ is spanned as a K-vector space by monomials*

(1) kv_i *with* $k \in K$ *and* $v_i \in E^0$, *or*
(2) $ke_1 \dots e_a f_1^* \dots f_b^*$ *where* $k \in K$; $a, b \geq 0$, $a + b > 0$, $e_1, \dots, e_a, f_1, \dots, f_b \in E^1$.

Lemma 2. *If* $\mu, \nu \in CSP(v)$, *then* $\mu^* \nu = \delta_{\mu,\nu} v$. *For every* $\mu \in CP(v)$ *there exist unique* $\mu_1, \dots, \mu_m \in CSP(v)$ *such that* $\mu = \mu_1 \cdots \mu_m$.

For a given graph E we define a preorder $\geq$ on the vertex set E^0 by: $v \geq w$ if and only if $v = w$ or there is a path μ such that $s(\mu) = v$ and $r(\mu) = w$. We say that a subset $H \subseteq E^0$ is *hereditary* if $w \in H$ and $w \geq v$ imply $v \in H$. We say H is *saturated* if whenever $0 < |s^{-1}(v)| < \infty$ and $\{r(e) : s(e) = v\} \subseteq H$, then $v \in H$. The *hereditary saturated closure* of a set $X \subset E^0$ is defined as the smallest

hereditary and saturated subset of E^0 containing X. For the hereditary saturated closure of X we use the notation given in [3]: $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

$$\Lambda_0(X) := \{v \in E^0 \mid x \geq v \text{ for some } x \in X\}, \text{ and for } n \geq 1,$$

$$\Lambda_n(X) := \{y \in E^0 \mid 0 < |s^{-1}(y)| < \infty \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X).$$

Example 3. Let $E = (E^0, E^1, r, s)$ be the directed graph where $E^0 = \{v, w\}$ and $E^1 = \{e_1, e_2, e_3\}$ such that $r(e_1) = s(e_1) = v$ and $r(e_2) = r(e_3) = s(e_3) = w$.

Then $\overline{\{v_1\}} = \{v_1, v_2\}$, whereas $\overline{\{v_2\}} = \{v_2\}$.

Example 4. Let $E = (E^0, E^1, r, s)$ be a directed graph where $E^0 = \{v_i \mid i \in \mathbb{Z}\}$ and $E^1 = \{e_i \mid i \in \mathbb{Z}\}$ such that $r(e_i) = v_i$ and $s(e_i) = v_{i-1}$.

Let $X = \{v_0\}$. Then we get $\Lambda_0(X) = \{v_0, v_1, \dots\}$. Furthermore,

$$\begin{aligned}\Lambda_1(X) &= \Lambda_0\{v_0\} \cup \{y \in E^0 \mid 0 < |s^{-1}(y)| < \infty \text{ and } r(s^{-1}(y)) \subseteq \Lambda_0(X)\} \\ &= \{v_0, v_1, \dots\} \cup \{v_{-1}\} \\ &= \{v_{-1}, v_0, v_1, \dots\}.\end{aligned}$$

Similarly, $\Lambda_k(X) = \{v_{-k}, v_{-k+1}, \dots\}$, and hence $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X) = E^0$.

With the introductory remarks now complete, we begin our discussion of the main result with the following important observation.

Remark 5. If I is a two-sided ideal of $L_K(E)$ and $\mu = \mu_1 + \cdots + \mu_n$ is in I, where $\mu_1, \dots, \mu_n$ are monomials in $L_K(E)$, then $\gamma_i = s(\mu_i)\mu r(\mu_i)$ is the sum of those μ_j whose sources are all the same and whose ranges are all the same; specifically, the sum of those μ_j for which $s(\mu_j) = s(\mu_i)$ and $r(\mu_j) = r(\mu_i)$. Moreover, $\gamma_i \in I$. Thus we may write $\mu = \gamma_1 + \cdots + \gamma_m$, with each γ_i with the above properties.

Notation. Let $L_K(E)_{\mathrm{R}}$ (resp., $L_K(E)_{\mathrm{G}}$) be the subring of elements in $L_K(E)$ whose terms involve only real edges (resp., only ghost edges).

Lemma 6. *Let I be a two-sided ideal of $L_K(E)$ and $I_{\mathrm{real}} = I \cap L_K(E)_{\mathrm{R}}$. Then I_{real} is the two-sided ideal of $L_K(E)_{\mathrm{R}}$ generated by elements of I_{real} having the form $v + \sum_{i=1}^{n} \lambda_i g^i$, where $v \in E^0$, g is a cycle based at v and $\lambda_i \in K$ for $1 \leq i \leq n$.*

Proof. Let J be the ideal of $L_K(E)_{\mathrm{R}}$ generated by elements in I_{real} of the indicated form. Our claim is $J = I_{\mathrm{real}}$. Towards a contradiction, suppose $I_{\mathrm{real}} \setminus J \neq \emptyset$; choose $\mu \in I_{\mathrm{real}} \setminus J$ of minimal length. By Remark 5, we can write $\mu = \tau_1 + \cdots + \tau_m$ with each τ_i is in I_{real} and is the sum of those paths whose sources are all the same and whose ranges are all the same. Since $\mu \notin J$, one of the $\tau_i \notin J$. Replacing μ by τ_i, we may assume that $\mu = \lambda_1\mu_1 + \cdots + \lambda_n\mu_n$ where all the μ_i have the same source and the same range. First we claim that one of the μ_i must have length 0, i.e. $\mu_i = kv$ for some vertex $v \in E^0$ and $k \in K$. Suppose not. Then for each i we can write $\mu_i = e_i\nu_i$ where $e_i \in E^1$. So $\mu = \sum_{i=1}^{n} e_i\nu_i$. Now

$$e_i^*\mu = \sum_{\{j \mid e_j = e_i\}} \lambda_j \nu_j \in I_{\mathrm{real}}$$

and has smaller length than μ. So $e_i^*\mu \in J$ and hence clearly $e_ie_i^*\mu \in J$. Then

$$\mu = \sum_{\text{distinct } e_i} e_ie_i^*\mu \in J,$$

a contradiction. So we can assume without loss of generality that $\mu_1 = kv$, with v a vertex. Since all the terms in μ have the same source and the same range, each μ_i is a closed path based at v. Multiplying by a scalar if necessary we can write $\mu = v + \lambda_2\mu_2 + \cdots + \lambda_n\mu_n$, μ_i closed paths at v.

Case I: There exists no, or exactly one, closed simple path at v. If there are no closed simple paths at v then we get $\mu \in J$, a contradiction. If there is exactly one closed simple path g based at v then necessarily g must be a cycle. Furthermore, the only paths in E which have source and range equal to v are powers of g. Then $\mu = v + \sum_{i=2}^{n} \lambda_i g^{m_i} \in J$, a contradiction.

Case II: There exist at least two distinct closed simple paths g_1 and g_2 based at v. As $g_1 \neq g_2$ and neither is a subpath of the other, $g_2^*g_1 = 0 = g_1^*g_2$. Without loss of generality assume $|\mu_2| \geq \cdots \geq |\mu_n| \geq 1$. Then for some $k \in \mathbb{N}$ $|g_1^k| > |\mu_2|$. Multiplying by $(g_1^*)^k$ on the left and g_1^k on the right, we get

$$\mu' = (g_1^*)^k\mu(g_1)^k = v + \sum_{i=2}^{n} \lambda_i (g_1^*)^k\mu_i(g_1)^k.$$

Note that if $0 \neq (g_1^*)^k\mu_i(g_1)^k$, then $(g_1^*)^k\mu_i \neq 0$. Since $|g_1^k| > |\mu_i|$, we get $g_1^k = \mu_i\mu_i'$ for some path μ_i'. Since the μ_i are closed paths based at the vertex v, one gets from the equation $(g_1)^k = \mu_i\mu_i'$ that $\mu_i = (g_1)^r$ for some integer $r \leq k$. So μ_i commutes with $(g_1)^k$ and thus each non-zero term $(g_1^*)^k\mu_i(g_1)^k = \mu_i$.

Since $g_2^*g_1 = 0$, $g_2^*\mu_i = 0$ for every $i \in \{2, \ldots, n\}$ and so we get $g_2^*\mu' g_2 = g_2^*vg_2 = v \in I \cap L_K(E)_{\mathrm{R}} = I_{\mathrm{real}}$, which implies that v is in J. Then $\mu = \mu v \in J$, a contradiction. □

It can be easily shown that the analogue of Lemma 6 is true for I_{ghost}. We state this for the sake of completeness.

Lemma 7. *Let I be a two-sided ideal of $L_K(E)$. Then I_{ghost} is generated by elements of the form $v + \sum_{k=1}^{m} \lambda_k(g^*)^k$, where $v \in E^0$, g is a cycle at v and $\lambda_i \in K$ for $1 \leq i \leq n$.*

Theorem 8. *Let E be a row-finite graph. Let I be any two-sided ideal of $L_K(E)$. Then I is generated by elements of the form $v + \sum_{k=1}^{m} \lambda_k g^k$, where $v \in E^0$, g is a cycle at v and $\lambda_1, \ldots, \lambda_m \in K$.*

Proof. Let J be the two-sided ideal of $L_K(E)$ generated by I_{real}. By Lemma 6, it is enough if we show that $I = J$. Suppose not. Choose $x = \sum_{i=1}^{d} \lambda_i\mu_i\nu_i^*$ in $I \setminus J$, where d is minimal and $\mu_1, \ldots, \mu_d, \nu_1, \ldots, \nu_d$ are real paths in $L_K(E)_R$. By Remark 5, $x = \alpha_1 + \cdots + \alpha_m$, where each $\alpha_i \in I$ and is a sum of those monomials all having the same source and same range. Since $x \notin J$, $\alpha_j \notin J$ for some j.

By the minimality of d, we can replace x by α_i. Thus we we can assume that $x = \sum_{i=1}^{d} \lambda_i \mu_i \nu_i^*$, where for all i, j, $s(\mu_i\nu_i^*) = s(\mu_j\nu_j^*)$ and $r(\mu_i\nu_i^*) = r(\mu_j\nu_j^*)$. Among all such $x = \sum_{i=1}^{d} \lambda_i \mu_i \nu_i^* \in I \setminus J$ with minimal d, select one for which $(|\nu_1|, \dots, |\nu_d|)$ is the smallest in the lexicographic order of $(\mathbb{Z}^+)^d$. First note that we have $|\nu_i| > 0$ for some i, otherwise x is in $I_{\text{real}} \subset J$. Let e be in E^1. Then note that

$$xe = \sum_{i=1}^{d} \lambda_i \mu_i \nu_i^* e = \sum_{i=1}^{d'} \lambda_i \mu_i (\nu_i')^*$$

either has fewer terms ($d' < d$), or $d = d'$ and $(|\nu_i'|, \dots, |\nu_d'|)$ is smaller than $(|\nu_1|, \dots, |\nu_d|)$. Then by minimality, we get xe is in J for every $e \in E^1$. Since $|\nu_i| > 0$ for some i, w is not a sink and emits finitely many edges. Hence we have

$$x = xw = x \sum_{\{e_j \in E^1 : s(e_j) = v\}} e_j e_j^* = \sum_{\{e_j \in E^1 : s(e_j) = v\}} (xe_j) e_j^* \in J.$$

We get a contradiction, so the result follows. □

Remark 9. We note that the Theorem does not hold for arbitrary graphs. An example is the "infinite clock": Let $E^0 = \{v, w_1, w_2, \dots\}$ and $E^1 = \{e_1, e_2, \dots\}$ with $r(e_i) = w_i$ and $s(e_i) = v$. Then the two-sided ideal generated by $v - e_1e_1^*$ is not generated by the elements of the desired form.

Notation. The element $v + \sum_{i=1}^{n} \lambda_i g^i$ is denoted by $p(g)$, where $p(x) = 1 + \lambda_2 x + \cdots + \lambda_n x^n \in K[x]$.

Now the Theorem is in hand, we are going to put the pieces together to get the Noetherian result.

Remark 10. Let g be a cycle based at $v \in E^0$ and let $p_1(x), p_2(x) \in K[x]$ be such that $p_1(g), p_2(g) \in I$. If we let $q(x) = \gcd(p_1(x), p_2(x)) \in K[x]$, then $q(g) \in < p_1(g), p_2(g) >$.

Lemma 11. *Let I be a two-sided ideal of $L_K(E)$, where E is an arbitrary graph. Suppose g, h are two non-trivial cycles based at distinct vertices u, v respectively. Suppose $u + \sum a_r g^r = p(g)$ and $v + \sum b_s h^s = q(h)$ both belong to I, where $p(x)$ and $q(x)$ are polynomials of smallest positive degree in $K[x]$ with $p(0) = 1 = q(0)$ such that $p(g) \in I$ and $q(h) \in I$. If $u \geq v$, then $q(h) \in < p(g) >$.*

Proof. Let μ be a path from u to v. We claim v must lie on the cycle g. Because, otherwise, $\mu^* g = 0$ and so $\mu^* p(g) \mu = \mu^* u \mu + \sum a_r \mu^* g^r \mu = v \in I$. This contradicts the fact that $\deg q(x) > 0$. So we can write $g = \mu\nu$ and $h = \nu\mu$ where ν is the part of g from v to u. Since $\mu^* g \mu = h$, we get $\mu^* p(g) \mu = p(h) \in I$. By the minimality of $q(x)$, $q(x)$ is a divisor of $p(x)$ in $K[x]$. Similarly, since $\nu^* q(h) \nu = q(g) \in I$, we conclude that $p(x)$ is a divisor of $q(x)$. Thus $q(x) = kp(x)$ for some $k \in K$. Since $p(0) = 1 = q(0)$, $q(x) = p(x)$. Hence $q(h) = p(h) = \mu^* p(g) \mu \in < p(g) >$. □

The next Lemma and its proof is implicit in the proof of Lemma 7 in [2].

Lemma 12. *Let E be an arbitrary graph and $S \subset E^0$. If $v \in \overline{S}$, and there is a non-trivial cycle based at v, then $u \geq v$ for some $u \in S$.*

Proof. We recall that $\overline{S} = \cup_{n\geq 0}\Lambda_n(S)$. Let k be the smallest non-negative integer such that $v \in \Lambda_k(S)$. We prove the Lemma by the induction on k, the Lemma being true by definition when $k = 0$. Assume $k > 0$ and that the Lemma holds when $k = n - 1$. Let $k = n$. Since $v \in \Lambda_n(S) \setminus \Lambda_{n-1}(S)$, $0 < |s^{-1}(v)| < \infty$ and $\{w_1, \ldots, w_m\} = r(s^{-1}(v)) \subset \Lambda_{n-1}(S)$. Since v is the base of a non-trivial cycle g, one of the vertices, say, w_j lies on the cycle g and so $w_j \geq v$. Since $w_j \in \Lambda_{n-1}(S)$ and is the base of a cycle, by induction there is a $u \in S$ such that $u \geq w_j$. Then $u \geq v$, as desired. $\square$

We also need the following Lemma, whose proof is given in the first paragraph of the proof of Theorem 5.7 in [8].

Lemma 13. *Let E be an arbitrary graph and let H be a hereditary and saturated subset of vertices in E. If I is the two-sided ideal generated by H, then $I \cap E^0 = H$.*

Theorem 14. *Let E be a row-finite graph. Then the following are equivalent:*

1. *$L_K(E)$ has a.c.c. on two-sided ideals,*
2. *$L_K(E)$ has a.c.c. on two-sided graded ideals,*
3. *The hereditary saturated closures of the subsets of the vertices in E^0 satisfy a.c.c..*

Proof. (3) $\Rightarrow$ (1) Suppose the ascending chain condition holds on the hereditary saturated closures of the subsets of E^0. Let I be a two-sided ideal of $L_K(E)$. By Theorem 8 and by Remark 10, I is generated by the set

$$\begin{aligned} T = \quad & \{v + \textstyle\sum_r \lambda_r g^r = p(g) \in I \mid v \in E^0, g \text{ is a cycle (may be trivial) based at } v \text{ and} \\ & p(x) \in K[x] \text{ is a polynomial of smallest degree such that } p(g) \in I \text{ and } p(0) = 1\}. \end{aligned}$$

It is well known that two-sided Noetherian is equivalent to every two-sided ideal being finitely generated, so we wish to show that I is generated by a finite subset of T.

Suppose, towards a contradiction, there are infinitely many $p_i(g_i) = v_i + \sum \lambda_r g_i^r \in T$ with $i \in H$, an infinite set and for each i, g_i is a non-trivial cycle based at v_i and that $\deg p_i(x) > 0$. By Lemma 11, we may assume that for any two i, j with $i \neq j$, $v_i \ngeq v_j$. Well-order the set H and consider it as the set of all ordinals less than an infinite ordinal κ. Define $S_1 = v_1$ and for any $\alpha < \kappa$, define $S_\alpha = \cup_{\beta<\alpha} S_\beta$ if α is a limit ordinal, and define $S_\alpha = S_\beta \cup \{v_{\beta+1}\}$ if α is a non-limit ordinal of the form $\beta + 1$. By the hypothesis the ascending chain of hereditary saturated closures of subsets $\overline{S}_1 \subset \overline{S}_2 \subset \cdots \overline{S}_\alpha \subset \cdots$ becomes stationary after a finite number of steps. So there is an integer n such that $\overline{S}_n = \overline{S}_{n+1} = \cdots$. Now

$v_{n+1} \in \overline{S}_{n+1} = \overline{S}_n$ and by Lemma 12, there is a $v_i \in S_n$ such that $v_i \geq v_{n+1}$. This is a contradiction. Hence the set $W = \{p_i(g_i) \in T \mid \deg p_i(x) > 0\}$ is finite.

So by the previous paragraph, if there are only finitely many $p_i(g_i)$ in T with $\deg p_i(x) = 0$, that is, only finitely many vertices in T, then we are done. We index the vertices v_α in T by ordinals $\alpha < \kappa$, an infinite ordinal, then as before, we get a well-ordered ascending chain of hereditary saturated closure of subsets $\overline{S}_1 \subset \overline{S}_2 \subset \cdots \subset \overline{S}_\alpha \subset \cdots$ $(\alpha < \kappa)$ where $S_1 = \{v_1\}$ and the S_α are inductively defined as before. Since, by hypothesis, this chain becomes stationary after a finite number of steps, there is an integer n such that $\overline{S}_\alpha = \overline{S}_n$ for all $\alpha > n$. Thus $\{v_\alpha \mid \alpha < \kappa\} \subset \overline{S}_n$. Since the ideal generated by the finite set $S_n = \{v_1, \ldots, v_n\}$ contains $\overline{S}_n$, we conclude that the ideal I is generated by the finite set $W \cup S_n$. Thus the Leavitt path algebra is two-sided Noetherian.

(1) $\Rightarrow$ (3) Conversely, suppose $L_K(E)$ is two-sided Noetherian. Consider an ascending chain of hereditary saturated closures of subsets of vertices $\overline{S}_1 \subset \overline{S}_2 \subset \cdots$ in E^0. Consider the corresponding ascending chain of two-sided ideals $I_1 \subset I_2 \subset \cdots$, where for each integer i, I_i is the two-sided ideal generated by $\overline{S}_i$. By hypothesis, there is an integer n such that $I_n = I_i$ for all $i > n$. We claim that $\overline{S}_i = \overline{S}_n$ for all $i > n$. Otherwise, we can find a vertex $w \in \overline{S}_i \setminus \overline{S}_n$ and since $w \in I_i = I_n$, $w \in I_n \cap E^0 = \overline{S}_n$ by Lemma 13 and this is a contradiction.

(1) $\Leftrightarrow$ (2) It is well-known (see [8]) that if $I(H)$ is a two-sided ideal of $L_K(E)$ generated by a hereditary and saturated subset H of E^0, then $I(H)$ is a graded ideal of $L_K(E)$. If we call $L_K(E)$ graded two-sided Noetherian if graded two-sided ideals of $L_K(E)$ satisfy the ascending chain condition, then Theorem 14 states that for any graph E, $L_K(E)$ is two-sided Noetherian if and only if it is graded two-sided Noetherian. □

Remark 15. We note that this result only shows that the a.c.c. on graded ideals is sufficient to get a.c.c. on all ideals, and that we are not proving that every ideal in a Noetherian Leavitt path algebra is graded. As an example we can consider $K[x, x^{-1}]$, which is the Leavitt path algebra of the graph with one vertex and one loop. Note that although this Leavitt path algebra has infinitely many ideals, it is nonetheless Noetherian, but has only the trivial graded ideals.

Now we easily see

Corollary 16. *Every Leavitt path algebra with a finite graph is two-sided Noetherian.*

We conclude by presenting another example of a non-Noetherian Leavitt path algebra.

Example 17. Let $E = (E^0, E^1, r, s)$ be the directed graph where $E^0 = \{v, w_1, w_2, w_3, \ldots\}$ and $E^1 = \{e_1, e_2, \ldots\} \cup \{f_1, f_2, \ldots\}$ is such that $r(e_i) = v$ and $s(e_i) = r(f_i) = s(f_i) = w_i$. The graph of this Leavitt path algebra is given in Figure 17. Note that if we let $S_i = \{w_1, \ldots, w_i\}$, then $\overline{S_1} \subset \overline{S_2} \subset \cdots$ is a non-

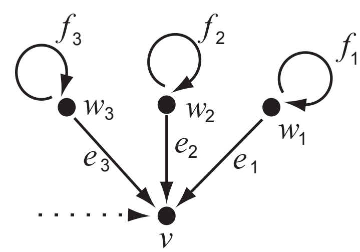

Fig. 1. Leavitt path algebra defined in Example 17.

terminating ascending chain of hereditary saturated closures of sets in E^0. Hence by Theorem 8, $L_K(E)$ is not two-sided Noetherian. Indeed, $< w_1 > \subset < w_1, w_2 > \subset \cdots$ is a non-terminating ascending chain of ideals in $L_K(E)$.

In [4] we present some additional consequences of Theorem 8, including a description of the two-sided artinian Leavitt path algebras.

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